ELSEVIER Linear Algebra and its Applications 302–303 (1999) 295–302
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

# A characterization of trace zero nonnegative 5×5 matrices

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Received 23 October 1998; accepted 22 May 1999

Submitted by M. Goldberg

Dedicated to Hans Schneider

**Abstract**

Let $\sigma = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ be a list of complex numbers with $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 0$. Necessary and sufficient conditions for the existence of an entry-wise nonnegative $5 \times 5$ matrix $A$ with spectrum $\sigma$ are presented. 

*Keywords:* Matrix; Spectrum; Graph; Cycles; Petersen graph; Nonnegative inverse eigenvalue problem

## 1. Introduction

Given a list $\sigma = (\lambda_1, \ldots, \lambda_n)$of complex numbers, the nonnegative inverse eigenvalue problem (NIEP) asks for the determination of necessary and sufficient conditions for the existence of an entrywise nonnegative $n \times n$ matrix $A$ with spectrum $\sigma$. In this paper the problem is solved for $n = 5$ in the "trace zero" case, that is, when $\lambda_1 + \cdots + \lambda_5 = 0$. Previous results on this problem for 'small' matrices include the case $n = 3$ (without further restriction) by Loewy and London [3], $n = 4$ and $\sigma$ a list of real numbers also by Loewy and London [3], $n = 4$ and $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$ by Reams [6]. Several other special cases of the NIEP for $n = 4$ have also been resolved by Reams [5]. For more results on the NIEP see [1,4].

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0024-3795/99/$ - see front matter 
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## 2. An inequality

We now consider the NIEP for $n = 5$ when the sum of the five complex numbers is zero. So let $\sigma = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\ )$ be a list of complex numbers. Define

$$s_k := \lambda_1^k + \lambda_2^k + \lambda_3^k + \lambda_4^k + \lambda_5^k, \quad k = 1, 2, 3, \ldots$$

and assume $s_1 = 0$. We wish to determine necessary and sufficient conditions so that there exists a nonnegative $5 \times 5$ matrix $A$ with spectrum $\sigma(A) = \sigma$. Suppose that such an $A$ exists. Then,

$$s_k = \mathrm{tr}\ (A^k) \quad \text{for } k = 1, 2, 3, \ldots$$

Note that since $A$ is entrywise nonegative, $s_k \geqslant 0$ for all $k$. Since $s_1 = 0$, all the diagonal entries of $A$ are zero. In this section we prove the following theorem.

**Theorem 2.1.** *Let $A$ be an entrywise nonnegative $5 \times 5$ matrix with* tr $(A) = 0$. *Then,*

$$12s_5 - 5s_2s_3 + 5s_3\sqrt{4s_4 - s_2^2} \geqslant 0.$$

**Proof.** The result is clear if $12s_5 \geqslant 5s_2s_3$, since $4s_4 - s_2^2 \geqslant 0$ by [2], so we first analyse the circumstances in which $12s_5 < 5s_2s_3$. Let $A = (a_{ij})$. Write $x_1, x_2, \ldots, x_{10}$ for the 2-cycle products $a_{12}a_{21}, a_{13}a_{31}, \ldots, a_{45}a_{54}$. So,

$$s_2 = 2(x_1 + x_2 + \cdots + x_{10}).$$

There are $\binom{5}{3} = 10$ 3-subsets $\{\alpha, \beta, \gamma\}$ of $\{1,2,3,4,5\}$ and each 3-subset of symbols gives rise to two 3-cycles in $s_3$. For example, $\{1,2,3\}$ gives the terms $a_{12}a_{23}a_{31}$ and $a_{13}a_{32}a_{21}$ and each occurs in $s_3$ with a multiplicity of 3. We can identify each 3-subset as the complement of the corresponding 2-subset. Denote these 3-cycle contributions by $y_1, y_2, \ldots, y_{10}$ where if $x_i = a_{rs}a_{sr}$, say, $y_i$ corresponds to the 3-cycles from the complement of $\{r, s\}$ in $\{1,2,3,4,5\}$. For example, if $x_1 = a_{12}a_{21}$ then $y_1 = a_{34}a_{45}a_{53} + a_{35}a_{54}a_{43}$. Note that

$$s_3 = 3(y_1 + y_2 + \cdots + y_{10}).$$

Thus,

$$5s_2s_3 = 30(x_1 + x_2 + \cdots + x_{10})(y_1 + y_2 + \cdots + y_{10}).$$

Next consider $s_5$. Since $A$ has a zero diagonal, $s_5$ is made up of contributions of the form (2-cycle) $\times$ (3-cycle) and 5-cycles. For example, $a_{12}a_{21}a_{14}a_{43}a_{31}$ and $a_{12}a_{23}a_{34}a_{45}a_{51}$ are terms occurring in the (1, 1) entry of $A^5$. The general (1, 1) entry of $A^5$ is $a_{1k}a_{kl}a_{lm}a_{mn}a_{n1}$ with $k \neq 1, k \neq l, l \neq m, m \neq n, n \neq 1$ (otherwise the term is zero). In this position, if the term is not a 5-cycle type term, then either l $= 1$ or $m = 1$. Note that a term of the form $a_{12}a_{21}a_{34}a_{45}a_{53}$ cannot occur in $s_5$. In terms of the $x$'s, and $y$'s previously defined, $x_1y_1, x_2y_2, \ldots, x_{10}y_{10}$ do not occur in $s_5$ (since by our choice of notation, the 3-cycles involved in $y_i$ are made from the symbols complementary to the symbols forming the 2-cycle in $x_i$). Hence in $12s_5 - 5s_2s_3$ we get the term $x_1y_1 + x_2y_2 + \cdots + x_{10}y_{10}$ with coefficient $-30$.

Consider now a product $x_iy_j$ where $i \neq j$. Suppose that $x_i$ corresponds to the 2-cycle term generated by the symbols $\{1, 2\}$, so $y_j$ corresponds to the terms generated by the symbols $\{p, q, r\}$, where $\{p, q, r\}$ contains at least one symbol of $\{1, 2\}$.

*Case* 1: $p = 1, q = 2$. Say $r = 3$. In this case $y_j = a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21}$. Note that $a_{12}a_{21}a_{12}a_{23}a_{31}$ and $a_{12}a_{21}a_{13}a_{32}a_{21}$ both occur in $s_5$ with coefficient 5 and thus these terms occur in $12s_5 - 5s_2s_3$ with coefficient $12 \times 5 - 30 = 30$.

*Case* 2: $\{1, 2\} \cap \{p, q, r\}$ contains one element. Suppose $p = 1$ is common and let $q = 3, r = 4$. We find that $s_5$ contains $a_{12}a_{21}a_{13}a_{34}a_{41}$ and $a_{12}a_{21}a_{14}a_{43}a_{31}$ both with coefficient 5. So these terms occur in $12s_5 - 5s_2s_3$ with coefficient $+30$.

The same arguments apply with symbols 1 and 2 replaced by $p$, and $q$, respectively. Hence,

$$12s_5 - 5s_2s_3 \geqslant 30\sum_{i=1}^{10}\sum_{j\neq i, j=1}^{10} x_iy_j - 30\sum_{i=1}^{10} x_iy_i$$

( $\geqslant$ because we have excluded the 5-cycles).

We are supposing that $12s_5 - 5s_2s_3 < 0$. Then,

$$x_1y_1 + x_2y_2 + \cdots + x_{10}y_{10} > \sum_{i=1}^{10}\sum_{j\neq i, j=1}^{10} x_iy_j. \tag{2.1}$$

Let $x_1$ be the maximum of $x_1, x_2, \ldots, x_{10}$ and suppose that

$$x_1 \leqslant x_2 + \cdots + x_{10}.$$

Now the right-hand side of (2.1) contains $x_1(y_2 + \cdots + y_{10}) + y_1(x_2 + \cdots + x_{10})$. But,

$$\begin{aligned} x_1(y_2 + \cdots + y_{10}) + y_1(x_2 + \cdots + x_{10}) &\geqslant x_1(y_2 + \cdots + y_{10}) + y_1x_1 \\ &\geqslant x_1y_1 + x_2y_2 + \cdots + x_{10}y_{10} \end{aligned}$$

since $x_2 \leqslant x_1, \ldots, x_{10} \leqslant x_1$. However, this says that the right- hand side of (2.1) is greater than or equal to the left-hand side which is false. Hence, we have proved.

*Step* 1: If $12s_5 < 5s_2s_3$ and $x_1$ is the maximum of $x_1, x_2, \ldots, x_{10}$ then

$$x_1 > x_2 + x_3 + \cdots + x_{10}. \tag{2.2}$$

We now analyse $4s_4 - s_2^2$ under the assumption (2.2). Recall [2] that the part of $4s_4 - s_2^2$ which does not involve 4-cycles can be written as

$$4(x_1 \cdots x_{10})(J - 2P)\begin{pmatrix} x_1 \\ \vdots \\ x_{10}, \end{pmatrix}$$

where $J$ is the matrix of all "ones" and $P$ is the adjacency matrix of the Petersen graph $P_{10}$. Note that we can label $x_1, \dots, x_{10}$ so that $P$ is

$$\begin{pmatrix} C + C^{\mathrm{T}} & I \\ I & C^2 + (C^{\mathrm{T}})^2, \end{pmatrix}$$

where

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now

$$\begin{aligned}
&(x_1 \cdots x_{10})(J - 2P)\begin{pmatrix} x_1 \\ \vdots \\ x_{10} \end{pmatrix} \\
&\quad = (x_1 + \cdots + x_{10})^2 - 4[x_1x_2 + x_1x_5 + x_1x_6 + x_2x_3 + x_2x_7 + x_3x_4 \\
&\qquad + x_3x_8 + x_4x_5 + x_4x_9 + x_5x_{10} + x_6x_8 + x_6x_9 + x_7x_9 + x_7x_{10} + x_8x_{10}] \\
&\quad = (x_1 - (x_2 + \cdots + x_{10}))^2 + 4x_1 - 4[x_2x_3 + x_2x_7 + x_3x_4 + x_3x_8 + x_4x_5 \\
&\qquad + x_4x_9 + x_5x_{10} + x_6x_8 + x_6x_9 + x_7x_9 + x_7x_{10} + x_8x_{10}].
\end{aligned}$$

Note that each negative term $x_px_q$ has either $p$ or $q$ in $\{3,4,7,8,9,10\}$ and all the negative terms arise in the expansion of

$$(x_2 + x_3 + \cdots + x_{10})(x_3 + x_4 + x_7 + x_8 + x_9 + x_{10}).$$

Since

$$x_1 > x_2 + x_3 + \cdots + x_{10}$$

we get that $x_1(x_3 + x_4 + x_7 + x_8 + x_9 + x_{10}) \geqslant$ the sum of the terms $x_px_q$ occurring with negative signs. Hence,

$$4s_4 - s_2^2 \geqslant 4(x_1 \cdots x_{10})(J - 2P)\begin{pmatrix} x_1 \\ \vdots \\ x_{10} \end{pmatrix} \geqslant 4[x_1 - (x_2 + \cdots + x_{10})]^2.$$

Hence we have established.

*Step* 2: If $x_1 > x_2 + \cdots + x_{10}$, then $4s_4 - s_2^2 \geqslant 4[x_1 - (x_2 + \cdots + x_{10})]^2$. (It is worth noting that the pairing off of the negative terms $x_px_q$ with positive terms $x_rx_s$ with $s \in \{3,4,7,8,9,10\}$ is crucial for the inequality. For example, if say $x_5x_6$ arose with a negative sign, the argument would fail. The pairing property is possible precisely because the Petersen graph $P_{10}$ has no 3-cycles. If one added an additional edge to $P_{10}$, it would have caused a 3-cycle to be created, so the argument would have then failed.) We now return to

$$Z := 12s_5 - 5s_2s_3 + 5s_3\sqrt{4s_4 - s_2^2}.$$

If $12s_5 < 5s_2s_3$ then (2.2) holds and, using Step 2, we obtain

$$\begin{aligned}
Z &\geqslant 30\sum_{i=1}^{10}\sum_{j=1,i\neq j}^{10} x_iy_j - 30\sum_{i=1}^{10} x_iy_i + 5s_3[2(x_1 - (x_2 + \cdots + x_{10}))] \\
&= 30(x_1 + x_2 + \cdots + x_{10})(y_1 + y_2 + \cdots + y_{10}) \\
&\quad - 60(x_1y_1 + x_2y_2 + \cdots + x_{10}y_{10}) \\
&\quad + 30(x_1 - x_2 - \cdots - x_{10})(y_1 + y_2 + \cdots + y_{10}) \\
&= 60x_1(y_1 + y_2 + \cdots + y_{10}) - 60(x_1y_1 + x_2y_2 + \cdots + x_{10}y_{10}) \\
&\geqslant 0,
\end{aligned}$$

since $x_1y_i \geqslant x_iy_i$ for $i = 1, 2, \ldots, 10$. So $Z \geqslant 0$ for all trace zero nonnegative $5 \times 5$ matrices. This concludes the proof of the theorem.

## 3. Main results

The following theorem completely solves the NIEP for $n = 5$ when the sum of the five eigenvalues is zero.

**Theorem 3.1** (Main theorem). *Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and $\lambda_5$ be complex numbers and let $s_k = \lambda_1^k + \lambda_2^k + \lambda_3^k + \lambda_4^k + \lambda_5^k$ for $k = 1, 2, 3, \ldots$ Assume $s_1 = 0$. Then $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ is the spectrum of a nonnegative $5 \times 5$ matrix if and only if the following conditions hold*:

1. $s_k \geqslant 0$ *for* $k = 2, 3, 4, 5$
2. $4s_4 \geqslant s_2^2$, *and*
3. $12s_5 - 5s_2s_3 + 5s_3\sqrt{4s_4 - s_2^2} \geqslant 0.$

**Proof.** By [2] and Theorem 2.1 above, the conditions are necessary. So, suppose that the conditions hold. Consider

$$A := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ q & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ v & u & t & s & 0 \end{pmatrix}.$$

The characteristic polynomial of $A$ is

$$x^5 - (q+s)x^3 - tx^2 + (qs-u)x + qt - v.$$

The Newton identities state that

$$ka_{n-k} + s_1 a_{n-k+1} + s_2 a_{n-k+2} + \cdots + s_k a_n = 0$$

for $k = 1, 2, \ldots, n$ where $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with $a_n = 1$ is a monic polynomial with zeroes $\lambda_1, \lambda_2, \ldots, \lambda_n$ including multiplicities and $s_k = \lambda_1^k + \lambda_2^k + \cdots + \lambda_n^k$. Applying these identities for $k = 1, 2, \ldots, 5$ to the characteristic polynomial of $A$, we require

$$q + s = s_2/2. \tag{3.1}$$

$$t = s_3/3. \tag{3.2}$$

$$qs - u = \left(\frac{s_2^2}{2} - s_4\right)/4. \tag{3.3}$$

$$qt - v = \frac{s_2 s_3}{6} - \frac{s_5}{5}. \tag{3.4}$$

Now, Eqs. (3.1) and (3.2) reduce to the inequalities

$$qs \geqslant \frac{s_2^2}{8} - \frac{s_4}{4},$$

$$q\frac{s_3}{3} \geqslant \frac{s_2 s_3}{6} - \frac{s_5}{5} = \frac{5s_2s_3 - 6s_5}{30}.$$

Take $q = s = s_2/4$. Then the inequalities hold if

$$\frac{s_2 s_3}{12} \geqslant \frac{5s_2s_3 - 6s_5}{30},$$

that is, if $12s_5 \geqslant 5s_2s_3$. Then taking $\mu = (4s_4 - s_2^2)/16$, $v = (12s_5 - 5s_2s_3)/60$, the resulting matrix

$$A := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{s_2}{4} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{12s_5 - 5s_2s_3}{60} & \frac{4s_4 - s_2^2}{16} & \frac{s_3}{3} & \frac{s_2}{4} & 0 \end{pmatrix}$$

is nonnegative and has the given spectrum.

In the general case, we require

$$q + s = s_2/2,$$

$$qs \geqslant \frac{s_2^2}{8} - \frac{s_4}{4},$$

$$qs_3 \geqslant \frac{5s_2s_3 - 6s_5}{10}.$$

Write $s = s_2/2 - q$ and substitute into the second inequality to get

$$\frac{s_2q}{2} - q^2 \geqslant \frac{s_2^2}{8} - \frac{s_4}{4}.$$

Completing the square we get

$$(q - \frac{s_2}{4})^2 \leqslant \frac{s_4}{4} - \frac{s_2^2}{16} = \frac{1}{16}(4s_4 - s_2^2).$$

The choice

$$q = \frac{s_2}{4} + \frac{1}{4}\sqrt{4s_4 - s_2^2}$$

gives

$$qs_3 = \frac{1}{4}(s_2s_3 + s_3\sqrt{4s_4 - s_2^2}).$$

Hence,

$$qs_3 \geqslant \frac{1}{10}(5s_2s_3 - 6s_5),$$

since $12s_5 - 5s_2s_3 + 5s_3\sqrt{4s_4 - s_2^2} \geqslant 0$. This choice of $q$ thus satisfies the conditions provided $s \geqslant 0$. That is, we require

$$\frac{1}{4}\sqrt{4s_4 - s_2^2} \leqslant \frac{s_2}{4}$$

(since $q + s = s_2/2$ ) and thus that $2s_4 \leqslant s_2^2$. In this case

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{s_2}{4} + \frac{1}{4}\sqrt{4s_4 - s_2^2} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{12s_5 - 5s_2s_3 + 5s_3\sqrt{4s_4 - s_2^2}}{60} & 0 & \frac{s_3}{3} & \frac{s_2}{4} - \frac{1}{4}\sqrt{4s_4 - s_2^2} & 0 \end{pmatrix}$$

is a nonnegative matrix with the given spectrum. Suppose finally that $2s_4 > s_2^2$. Consider

$$A := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{s_2}{2} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{s_5}{5} & \frac{2s_4 - s_2^2}{8} & \frac{s_3}{3} & 0 & 0 \end{pmatrix}.$$

Then $A$ is nonnegative and has the desired spectrum. This concludes the proof of the theorem.

## Acknowledgements

We wish to thank Raphi Loewy for his helpful comments and especially for carefully reading an earlier draft of this paper.

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